Representations for weighted Moore-Penrose inverses of partitioned adjointable operators

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Abstract

For two positive definite adjointable operators M and N, and an adjointable operator A acting on a Hilbert C^*-module, some properties of the weighted Moore-Penrose inverse $A^\dagger_{MN}$ are established. If $A=(A_{ij})$ is 1×2 or 2×2 partitioned, then general representations for $A^\dagger_{MN}$ in terms of the individual blocks of A_{ij} are studied. In the case when A is 1×2 partitioned, a unified representation for $A^\dagger_{MN}$ is presented. In the 2×2 partitioned case, an approach to the construction of the Moore-Penrose inverse from the non-weighted case to the weighted case is provided. Some results known for matrices are extended to the general setting of operators on Hilbert C^*-modules.

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Introduction

The weighted Moore-Penrose inverse of an arbitrary (singular and rectangular) matrix has many applications in the weighted linear least-squares problems, statistics, neural network, numerical analysis and so on. For a partitioned matrix $A=(A_{ij})$, it has been of interest to derive general expressions for the weighted Moore-Penrose inverse of A in terms of the individual blocks of A_{ij}. If $A=(A_{11},A_{12})$ is a 1×2 partitioned matrix, then some formulas for the (non-weighted) Moore-Penrose inverse $A^\dagger$, such as Cline [2] and Mihalyffy [8] are well-known. In the weighted case, a formula for $A^\dagger_{MN}$ of a 1×2 partitioned matrix A was given by Miao [6]. Later, this formula was reproved by Chen [1], Wang and Zheng [10] by using different methods. Recently, another formula for $A^\dagger_{MN}$ has been obtained by the first author [11]. In this paper, in the general context of Hilbert C^*-module operators, we will provide a unified representation for $A^\dagger_{MN}$ (see Theorem 3.4 below). As a result, the equivalence of the formulas for $A^\dagger_{MN}$ given respectively in [6] and [11] is derived.

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If $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is a 2×2 partitioned matrix, then things may become much more complicated. Most works in literature concerning representations for $A^\dagger$ were carried out under certain restrictions on the blocks of A_{ij}. In 1991, a general expression for $A^\dagger$ without any restriction imposed on the blocks of A_{ij}, was given by Miao in [7]. Since then, more than twenty years has passed. However, due to the complexity revealed in [3, 5, 7], there has not been much progress concerning the generalization of Miao's result [7] from the non-weighted case to the weighted case. In this paper, we make such an effort in the general setting of Hilbert C^*-module operators.

The paper is organized as follows. In Section 1, in the general setting of Hilbert C^*-module operators, we will establish some properties on weighted Moore-Penrose inverses. Following the line initiated in [12], in Section 2 we will study the relationship between weighted Moore-Penrose inverses $A^\dagger_{MN}$, where A is fixed, while M and N are variable. In Section 3 we will study unified representations for weighted Moore-Penrose inverses of 1×2 partitioned adjointable operators. In Section 4, an approach, initiated in [11] for 1×2 partitioned adjointable operators, is applied to study the general expressions for weighted Moore-Penrose inverses of 2×2 partitioned adjointable operators. Our key point is the construction of a commutative diagram in page 16, through which the main results of [7] are generalized from the non-weighted case to the weighted case.

1 Weighted Moore-Penrose inverses of adjointable operators

In this section, in a general setting of adjointable operators on Hilbert C^*-modules, we establish some properties on weighted Moore-Penrose inverses, most of which are known for matrices. Throughout this paper, $\mathfrak{A}$ is a C^*-algebra, $\mathbb{C}$ is the complex field, and $\mathbb{C}^{m\times n}$ is the set of $m \times n$ complex matrices. By a projection, we mean an idempotent and a self-adjoint element in a certain C^*-algebra. For any Hilbert $\mathfrak{A}$-modules H and K, let $\mathcal{L}(H,K)$ be the set of *adjointable* operators from H to K. If $H = K$, then $\mathcal{L}(H,H)$, which we abbreviate to $\mathcal{L}(H)$, is a unital C^*-algebra, whose unit is denoted by I_H. For any $A \in \mathcal{L}(H,K)$, the range and the null space of A are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively.

Throughout, the notations of "$\oplus$" and "$\dotplus$" are used with different meanings. For any Hilbert $\mathfrak{A}$-modules H_1 and H_2, let

$$H_1 \oplus H_2 = \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \middle| \, h_i \in H_i, i = 1, 2 \right\},$$

which is also a Hilbert $\mathfrak{A}$-module whose $\mathfrak{A}$-valued inner product is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \text{ for any } x_i \in H_1 \text{ and } y_i \in H_2, i = 1, 2.$$

If both H_1 and H_2 are submodules of a Hilbert $\mathfrak{A}$-module H such that $H_1 \cap H_2 = \{0\}$, then we define

$$H_1 \dotplus H_2 = \{h_1 + h_2 \,|\, h_i \in H_i, i = 1, 2\} \subseteq H.$$

If furthermore $H = H_1 \dotplus H_2$, then we call P_{H_1,H_2} the *oblique projector* along H_2 onto H_1, where P_{H_1,H_2} is defined by

$$P_{H_1,H_2}(h) = h_1, \text{ for any } h = h_1 + h_2 \in H \text{ with } h_i \in H_i, i = 1, 2.$$

Lemma 1.1. (cf. [4, Theorem 3.2] and [13, Remark 1.1]) *Let H, K be two Hilbert $\mathfrak{A}$-modules and $A \in \mathcal{L}(H, K)$. Then the closeness of any one of the following sets implies the closeness of the remaining three sets:*

$$\mathcal{R}(A),\ \mathcal{R}(A^*),\ \mathcal{R}(AA^*), \mathcal{R}(A^*A).$$

Furthermore, if $\mathcal{R}(A)$ is closed, then $\mathcal{R}(A) = \mathcal{R}(AA^*)$, $\mathcal{R}(A^*) = \mathcal{R}(A^*A)$ and with respect to the $\mathfrak{A}$-valued inner product, the following orthogonal decompositions hold:

$$H = \mathcal{N}(A) \dotplus \mathcal{R}(A^*),\ K = \mathcal{R}(A) \dotplus \mathcal{N}(A^*). \tag{1.1}$$

Throughout the rest of this section, H, K and L are three Hilbert $\mathfrak{A}$-modules.

Definition 1.1. An element M of $\mathcal{L}(K)$ is said to be *positive definite*, if M is positive and invertible in $\mathcal{L}(K)$.

Proposition 1.2. *Let $M \in \mathcal{L}(K)$ be positive definite. Then with the inner-product given by*

$$\big\langle x, y\big\rangle_M = \big\langle x, My\big\rangle, \text{ for any } x, y \in K, \tag{1.2}$$

K also becomes a Hilbert $\mathfrak{A}$-module.

Proof. With respect to $\big\langle \cdot, \cdot\big\rangle_M$, K is clearly an inner-product $\mathfrak{A}$-module [4, P. 2]. We prove that K is complete with respect to the norm induced by

$$\|x\|_M \overset{def}{=} \|\big\langle x, x\big\rangle_M\|^{\frac{1}{2}} = \|M^{\frac{1}{2}}x\|, \text{ for any } x \in K. \tag{1.3}$$

In fact, if we let $C_1 = \|M^{-\frac{1}{2}}\|^{-1} > 0$ and $C_2 = \|M^{\frac{1}{2}}\| > 0$, then by (1.3) we can get

$$C_1\ \|x\| \leq\ \|x\|_M\ \leq C_2\ \|x\|, \text{ for any } x \in K,$$

which means that $\|\cdot\|$ and $\|\cdot\|_M$ are equivalent norms on K. Since K is assumed to be complete with respect to the original norm $\|\cdot\|$, the completeness of K with respect to the induced norm $\|\cdot\|_M$ follows. $\square$

Remark 1.1. We use the notation K_M to denote the Hilbert $\mathfrak{A}$-module with the inner-product given by (1.2), and call K_M the *weighted space* (with respect to M). Following the notation, for any positive definite element N of $\mathcal{L}(H)$, $T \in \mathcal{L}(H, K)$, $x \in H$ and $y \in K$, we have

$$\big\langle Tx, y\big\rangle_M = \big\langle Tx, My\big\rangle = \big\langle x, T^*My\big\rangle = \big\langle x, N^{-1}T^*My\big\rangle_N.$$

So if we regard T as an element of $\mathcal{L}(H_N, K_M)$[1], then

$$T^\# = N^{-1}\, T^*\, M, \tag{1.4}$$

where $T^\# \in \mathcal{L}(K_M, H_N)$ is the adjoint operator of $T \in \mathcal{L}(H_N, K_M)$.

[1] The reader should be aware that as sets, $\mathcal{L}(H, K)$ and $\mathcal{L}(H_N, K_M)$ are the same.

Definition 1.2. Let $A \in \mathcal{L}(H, K)$ be arbitrary, and let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be two positive definite operators. The *weighted Moore-Penrose inverse* $A^\dagger_{MN}$ (if it exists) is the element X of $\mathcal{L}(K, H)$, which satisfies

$$AXA = A, XAX = X, (MAX)^* = MAX \text{ and } (NXA)^* = NXA. \tag{1.5}$$

If $M = I_K$ and $N = I_H$, then $A^\dagger_{MN}$ is denoted simply by $A^\dagger$, which is called the *Moore-Penrose inverse* of A.

Theorem 1.3. *Let $A \in \mathcal{L}(H, K)$ be arbitrary, and let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be two positive definite operators. Then $A^\dagger_{MN}$ exists if and only if A has a closed range.*

Proof. If $A^\dagger_{MN}$ exists, then $AA^\dagger_{MN}$ is an idempotent, so $\mathcal{R}(A) = \mathcal{R}(AA^\dagger_{MN})$ is closed. Conversely, suppose that $\mathcal{R}(A)$ is closed in K, then $\mathcal{R}(A)$ is also closed in K_M, so by [13, Theorem 2.2 and Proposition 2.4] there exists uniquely an element $X \in \mathcal{L}(K_M, H_N)$ satisfying

$$AXA = A, \ XAX = X, (AX)^\# = AX \text{ and } (XA)^\# = XA.$$

By (1.4) we get $(AX)^\# = M^{-1}(AX)^*M$ and $(XA)^\# = N^{-1}(XA)^*N$. As M and N are self-adjoint, the last two equalities in (1.5) hold. □

Remark 1.2. Let $A \in \mathcal{L}(H, K)$ have a closed range, and let $M \in \mathcal{L}(K)$, $N \in \mathcal{L}(H)$ be positive definite. As in the finite-dimensional case [9, Theorem 1.4.4], by [13, Theorem 2.2] we have

$$\begin{aligned}\mathcal{R}(A^\dagger_{MN}) = \mathcal{R}(A^\#) = \mathcal{R}(N^{-1}A^*M) = N^{-1}\mathcal{R}(A^*),\\ \mathcal{N}(A^\dagger_{MN}) = \mathcal{N}(A^\#) = \mathcal{N}(N^{-1}A^*M) = M^{-1}\mathcal{N}(A^*).\end{aligned}$$

Proposition 1.4. (cf. [1, Lemma 0.1]) *Let $A \in \mathcal{L}(H, K)$ have a closed range, and let $M \in \mathcal{L}(K)$ and $N \in \mathcal{L}(H)$ be positive definite. Then $A^\dagger_{MN}$ is the unique element X of $\mathcal{L}(K, H)$ which satisfies*

$$A^*MAX = A^*M, \ \mathcal{R}(NX) \subseteq \mathcal{R}(A^*). \tag{1.6}$$

Proof. By [13, Proposition 2.4] we know that $A^\dagger_{MN}$ is the unique element X of $\mathcal{L}(K_M, H_N)$ which satisfies

$$AX = AA^\dagger_{MN} \text{ and } \mathcal{R}(X) \subseteq \mathcal{R}(A^\#). \tag{1.7}$$

In view of (1.4), we know that (1.6) can be rewritten as

$$A^\#AX = A^\#, \ \mathcal{R}(X) \subseteq R\big(A^\#\big). \tag{1.8}$$

Since $A^\#AA^\dagger_{MN} = A^\#$ and $\big(A^\dagger_{MN}\big)^\# A^\#AX = (AA^\dagger_{MN})^\#AX = AA^\dagger_{MN}AX = AX$, the equivalence of (1.7) and (1.8) follows. □

Lemma 1.5. (cf. [1, Lemma 0.3]) *Let $A \in \mathcal{L}(H, K)$ have a closed range, and let $M \in \mathcal{L}(K)$ be positive definite. Then for any $X \in \mathcal{L}(K, H)$, the following two statements are equivalent:*

(i) $AXA = A, (MAX)^* = MAX;$

(ii) $A^*MAX = A^*M$.

If condition (i) *is satisfied, then for any positive definite element $N \in \mathcal{L}(H)$, X has the form*

$$X = A^\dagger_{MN} + (I_H - A^\dagger_{MN}A)Y, \text{ for some } Y \in \mathcal{L}(K,H). \tag{1.9}$$

Proof. (1) Let N be any positive definite element of $\mathcal{L}(H)$. By (1.4) we know that conditions (i) and (ii) can be rephrased respectively as

$$AXA = A,\ (AX)^\# = AX, \tag{1.10}$$

$$A^\# AX = A^\#. \tag{1.11}$$

Suppose that (1.10) is satisfied. Then

$$A^\# AX = A^\#(AX)^\# = (AXA)^\# = A^\#.$$

Conversely, if (1.11) is satisfied, then it is easy to show that $(AXA - A)^\#(AXA - A) = 0$, so $AXA = A$. Furthermore,

$$(AX)^\# = X^\# A^\# = X^\# A^\# AX = (A^\# AX)^\# X = (A^\#)^\# X = AX.$$

(2) Suppose that $X \in \mathcal{L}(K,H)$ is given such that (1.11) is satisfied. Then

$$A^\# A(X - A^\dagger_{MN}) = A^\# - A^\# = 0 \Longrightarrow \big(A(X - A^\dagger_{MN})\big)^\# A(X - A^\dagger_{MN}) = 0,$$

so $A(X - A^\dagger_{MN}) = 0$; or equivalently, $A^\dagger_{MN}A(X - A^\dagger_{MN}) = 0$, hence there exists $Y \in \mathcal{L}(K,H)$ such that $X - A^\dagger_{MN} = (I_H - A^\dagger_{MN}A)Y$. □

Definition 1.3. An element X of $\mathcal{L}(K,H)$ is said to be a $(1,3)$-inverse of $A \in \mathcal{L}(H,K)$, written $X \in A\{1,3\}$, if $AXA = A$ and $(AX)^* = AX$.

Proposition 1.6. *Let $A \in \mathcal{L}(H,K)$ have a closed range. Then for any $X \in (AA^*)\{1,3\}$, we have $A^\dagger = A^*X$.*

Proof. Put $Y = A^*X$. By (1.6) it is sufficient to verify that

$$A^*AY = A^*,\ \mathcal{R}(Y) \subseteq \mathcal{R}(A^*).$$

The second condition is obviously satisfied. Replacing A, M with AA^* and I_K respectively, by "(i)$\Longrightarrow$ (ii)" in Lemma 1.5 we obtain $AA^*AA^*X = AA^*$, therefore

$$A^*AY = A^*AA^*X = A^\dagger(AA^*AA^*X) = A^\dagger AA^* = A^*. \quad \square$$

2 Relationship between weighted Moore-Penrose inverses

Throughout this section, H and K are two Hilbert $\mathfrak{A}$-modules, $M \in \mathcal{L}(K)$ and $N_1, N_2 \in \mathcal{L}(H)$ are three positive definite operators. The purpose of this section is to generalize [12, Lemma 2.4] from the finite-dimensional case to the Hilbert C^*-module case. For any $A \in \mathcal{L}(H,K)$, if $\mathcal{R}(A)$ is closed, then as in [12] we define

$$\begin{aligned} R_{M;N_1,N_2} &= I_H + (I_H - A^\dagger_{MN_1}A)N_1^{-1}(N_2 - N_1) \\ &= A^\dagger_{MN_1}A + (I_H - A^\dagger_{MN_1}A)N_1^{-1}N_2. \end{aligned} \tag{2.1}$$

Lemma 2.1. *Let $A \in \mathcal{L}(H,K)$ have a closed range. The operator $R_{M;N_1,N_2}$ defined by (2.1) is invertible.*

Proof. Let $P = A^\dagger_{MN_1}A$, $S = (I_H - P)N_1^{-1}N_2(I_H - P)$, $H_1 = (I_H - P)H$ and $S|_{H_1} : H_1 \to H_1$ be the restriction of S to H_1.

First, we prove that $S|_{H_1} \in \mathcal{L}(H_1)$ is invertible. By the last condition in (1.5) we get

$$N_1 S = (I_H - P)^* N_2 (I_H - P) = \big(N_2^{\frac{1}{2}}(I_H - P)\big)^* \big(N_2^{\frac{1}{2}}(I_H - P)\big). \tag{2.2}$$

As P is idempotent, we have $\mathcal{N}(S) = \mathcal{N}(N_1 S) = \mathcal{N}(I_H - P) = \mathcal{R}(P)$, which means that $\mathcal{N}(S|_{H_1}) = \mathcal{R}(P) \cap H_1 = \{0\}$. Furthermore, since $\mathcal{R}\big((N_2^{\frac{1}{2}}(I_H - P)\big)$ is closed, we may apply Lemma 1.1 to (2.2) to conclude that

$$\begin{aligned}\mathcal{R}(S|_{H_1}) &= \mathcal{R}(S) = N_1^{-1}\mathcal{R}(N_1 S) = N_1^{-1}\mathcal{R}\big((I_H - P)^* N_2^{\frac{1}{2}}\big)\\ &= \mathcal{R}\big(N_1^{-1}(I_H - P)^*\big) = \mathcal{R}\big((I_H - P)N_1^{-1}\big) = \mathcal{R}\big(I_H - P\big) = H_1.\end{aligned}$$

This completes the proof of the invertibility of $S|_{H_1}$.

Next, let

$$Y = P + (S|_{H_1})^{-1}(I_H - P) - (S|_{H_1})^{-1}(I_H - P)N_1^{-1}N_2 P.$$

Then since $R_{M;N_1,N_2} = P + (I_H - P)N_1^{-1}N_2 P + S$, it is easy to verify that $R_{M;N_1,N_2}Y = YR_{M;N_1,N_2} = I_H$. ☐

Lemma 2.2. (cf. [12, Lemma 2.4]) *Suppose that $A \in \mathcal{L}(H,K)$ has a closed range. Then $A^\dagger_{MN_2} = R^{-1}_{M;N_1,N_2}A^\dagger_{MN_1}$, where $R_{M;N_1,N_2}$ is defined by (2.1).*

Proof. Let $A^\# = N_1^{-1}A^*M \in \mathcal{L}(K_M, H_{N_1})$ be the conjugate operator of $A \in \mathcal{L}(H_{N_1}, K_M)$. To simplify the notation, we define

$$X = R^{-1}_{M;N_1,N_2} \cdot (I_H - A^\dagger_{MN_1}A)N_1^{-1}N_2. \tag{2.3}$$

Then

$$(I_H - A^\dagger_{MN_1}A)N_1^{-1}A^* = (I_H - A^\dagger_{MN_1}A)A^\# M^{-1} = 0,$$

so by (2.3) we have

$$XN_2^{-1}\mathcal{R}(A^*) = 0. \tag{2.4}$$

Since $A^\dagger_{MN_1}A(I_H - A^\dagger_{MN_1}A) = 0$, by (2.1) and (2.3) we have

$$\begin{aligned}X(I_H - A^\dagger_{MN_1}A) &= \big(R^{-1}_{M;N_1,N_2} \cdot A^\dagger_{MN_1}A + X\big)(I_H - A^\dagger_{MN_1}A)\\ &= R^{-1}_{M;N_1,N_2} \cdot \Big(A^\dagger_{MN_1}A + (I_H - A^\dagger_{MN_1}A)N_1^{-1}N_2\Big)(I_H - A^\dagger_{MN_1}A)\\ &= R^{-1}_{M;N_1,N_2} \cdot R_{M;N_1,N_2} \cdot (I_H - A^\dagger_{MN_1}A) = I_H - A^\dagger_{MN_1}A. \end{aligned}\tag{2.5}$$

As $I_H - A^\dagger_{MN_2}A$ is the oblique projector of H along $N_2^{-1}\mathcal{R}(A^*)$ onto $\mathcal{N}(A) = \mathcal{R}(I_H - A^\dagger_{MN_1}A)$, in view of (2.4) and (2.5) we conclude that $I_H - A^\dagger_{MN_2}A = X$. Furthermore, by (2.3) and (2.1) we have

$$\begin{aligned}I_H - A^\dagger_{MN_2}A &= X = R^{-1}_{M;N_1,N_2} \cdot (I_H - A^\dagger_{MN_1}A)N_1^{-1}N_2 \qquad (2.6)\\ &= R^{-1}_{M;N_1,N_2} \cdot (R_{M;N_1,N_2} - A^\dagger_{MN_1}A) = I_H - R^{-1}_{M;N_1,N_2} \cdot A^\dagger_{MN_1}A.\end{aligned}$$

It follows that

$$A^{\dagger}_{MN_2}A = R^{-1}_{M;N_1,N_2} \cdot A^{\dagger}_{MN_1}A. \tag{2.7}$$

Note that $AA^{\dagger}_{MN_1} = AA^{\dagger}_{MN_2}$ is the oblique projector of K along $M^{-1}\mathcal{N}(A^*)$ onto $\mathcal{R}(A)$, so if we multiply $A^{\dagger}_{MN_1}$ from the right on both sides of (2.7), then we may obtain

$$A^{\dagger}_{MN_2} = A^{\dagger}_{MN_2}AA^{\dagger}_{MN_2} = A^{\dagger}_{MN_2}AA^{\dagger}_{MN_1} = R^{-1}_{M;N_1,N_2} \cdot A^{\dagger}_{MN_1}. \quad \square$$

Remark 2.1. With the notation of Lemma 2.2, by (2.6) we obtain

$$(I_H - A^{\dagger}_{MN_2}A)N_2^{-1} = R^{-1}_{M;N_1,N_2} \cdot (I_H - A^{\dagger}_{MN_1}A)N_1^{-1}. \tag{2.8}$$

3 Unified representations for weighted Moore-Penrose inverses of 1×2 partitioned operators

Throughout this section, H_1, H_2 and H_3 are three Hilbert $\mathfrak{A}$-modules, $A \in \mathcal{L}(H_1, H_3)$ and $B \in \mathcal{L}(H_2, H_3)$ are arbitrary, $M \in \mathcal{L}(H_3)$ and

$$N = \begin{pmatrix} N_1 & L \\ L^* & N_2 \end{pmatrix} \in \mathcal{L}(H_1 \oplus H_2) \tag{3.1}$$

are two positive definite operators, where $N_1 \in \mathcal{L}(H_1), L \in \mathcal{L}(H_2, H_1)$ and $N_2 \in \mathcal{L}(H_2)$. By [11, Section 5] we know that both N_1 and $S(N)$ are positive definite, where $S(N)$ is the Schur complement of N defined by

$$S(N) = N_2 - L^* N_1^{-1} L.$$

When A has a closed range, we put

$$C = (I_{H_3} - AA^{\dagger}_{MN_1})B \in \mathcal{L}(H_2, H_3). \tag{3.2}$$

Lemma 3.1. *Let $A \in \mathcal{L}(H_1, H_3)$ have a closed range. Then*

(i) $\mathcal{R}\binom{A^*}{C^*} = \mathcal{R}(A^*) \oplus \mathcal{R}(C^*)$;

(ii) $\mathcal{R}(A^*) = \mathcal{N}\big((I_{H_1} - A^{\dagger}_{MN_1}A)N_1^{-1}\big)$.

Proof. (i) For any $\xi, \eta \in H_3$, let $\zeta = (AA^{\dagger}_{MN_1})^*\xi + (I_{H_3} - AA^{\dagger}_{MN_1})^*\eta$. Then $A^*\zeta = A^*\xi$ and $C^*\zeta = C^*\eta$, so $\binom{A^*\xi}{C^*\eta} = \binom{A^*}{C^*}\zeta \in \mathcal{R}\binom{A^*}{C^*}$.

(ii) As $AA^{\dagger}_{MN_1}A = A$, we have

$$\begin{aligned}\mathcal{R}(A^*) &= \mathcal{R}\big((A^{\dagger}_{MN_1}A)^*\big) = \mathcal{N}\big(I_{H_1} - (A^{\dagger}_{MN_1}A)^*\big) \\ &= \mathcal{N}\big(N_1(I_{H_1} - A^{\dagger}_{MN_1}A)N_1^{-1}\big) = \mathcal{N}\big((I_{H_1} - A^{\dagger}_{MN_1}A)N_1^{-1}\big). \quad \square\end{aligned}$$

Although the technique lemma in [1] ([1, Lemma 0.2]) is no longer true in the infinite-dimensional case, we can still provide a formula for $(A, C)^{\dagger}_{MN}$ by following the line in [1] together with some modifications.

Theorem 3.2. (cf. [1, Theorem 1.1]) *Let C be defined by (3.2), and suppose that $\mathcal{R}(A)$, $\mathcal{R}(C)$ and $\mathcal{R}(A,C)$ are all closed. Then*

$$(A,C)^\dagger_{MN} = \begin{pmatrix} A^\dagger_{MN_1} - (I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}LU \\ U \end{pmatrix}, \tag{3.3}$$

where

$$S = N_2 - L^*(I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}L = S(N) + L^*A^\dagger_{MN_1}AN_1^{-1}L \in \mathcal{L}(H_2), \tag{3.4}$$

$$U = C^\dagger_{MS} - (I_{H_2} - C^\dagger_{MS}C)S^{-1}L^*A^\dagger_{MN_1} \in \mathcal{L}(H_3, H_2). \tag{3.5}$$

Proof. Note that $A^\dagger_{MN_1}A$ is a projection on the weighted space $(H_1)_{N_1}$, so for any $\xi \in H_2$, we have

$$\langle L^*A^\dagger_{MN_1}AN_1^{-1}L\xi, \xi\rangle = \langle (A^\dagger_{MN_1}A)(N_1^{-1}L\xi), N_1^{-1}L\xi\rangle_{N_1} \geq 0,$$

hence $L^*A^\dagger_{MN_1}AN_1^{-1}L$ is positive [4, Lemma 4.1], which means that the operator S defined by (3.4) is positive definite. Note also that

$$C^*MA = B^*(I_{H_3} - AA^\dagger_{MN_1})^*MA = B^*M(I_{H_3} - AA^\dagger_{MN_1})A = 0. \tag{3.6}$$

Now let N_3 be any positive definite element of $\mathcal{L}(H_2)$. For any $X_1 \in \mathcal{L}(H_3, H_1)$ and $X_2 \in \mathcal{L}(H_3, H_2)$, by Proposition 1.4 we know that $\binom{X_1}{X_2} = (A,C)^\dagger_{MN}$ if and only if

$$\begin{pmatrix} A^* \\ C^* \end{pmatrix} M(A,C)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} A^* \\ C^* \end{pmatrix} M, \ R\left(N\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\right) \subseteq R\begin{pmatrix} A^* \\ C^* \end{pmatrix}. \tag{3.7}$$

Combining the above two conditions with (3.6), we may apply Lemma 3.1 to conclude that $\binom{X_1}{X_2} = (A,C)^\dagger_{MN}$ if and only if the following four equations hold:

$$A^*MAX_1 = A^*M; \tag{3.8}$$

$$C^*MCX_2 = C^*M; \tag{3.9}$$

$$(I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}(N_1X_1 + LX_2) = 0; \tag{3.10}$$

$$(I_{H_2} - C^\dagger_{MN_3}C)N_3^{-1}(L^*X_1 + N_2X_2) = 0. \tag{3.11}$$

By (1.9) we have

$$X_1 = A^\dagger_{MN_1} + (I_{H_1} - A^\dagger_{MN_1}A)Y_1, \tag{3.12}$$

$$X_2 = C^\dagger_{MN_3} + (I_{H_2} - C^\dagger_{MN_3}C)Y_2, \tag{3.13}$$

for some $Y_1 \in \mathcal{L}(H_3, H_1)$ and $Y_2 \in \mathcal{L}(H_3, H_2)$. It follows from (3.10) and (3.12) that

$$(I_{H_1} - A^\dagger_{MN_1}A)Y_1 + (I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}LX_2 = 0.$$

Combining the above equality with (3.12) we get

$$X_1 = A^\dagger_{MN_1} - (I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}LX_2. \tag{3.14}$$

It follows from (3.11), (3.14) and (3.4) that

$$(I_{H_2} - C^{\dagger}_{MN_3}C)N_3^{-1}L^*A^{\dagger}_{MN_1} + (I_{H_2} - C^{\dagger}_{MN_3}C)N_3^{-1}SX_2 = 0. \tag{3.15}$$

By (3.13) we have

$$\begin{aligned}(I_{H_2} - C^{\dagger}_{MN_3}C)N_3^{-1}SX_2 = (I_{H_2} - C^{\dagger}_{MN_3}C)N_3^{-1}SC^{\dagger}_{MN_3} \\ +(I_{H_2} - C^{\dagger}_{MN_3}C)N_3^{-1}S(I_{H_2} - C^{\dagger}_{MN_3}C)Y_2.\end{aligned} \tag{3.16}$$

So, if we let $N_3 = S$, then by the above equality we get

$$(I_{H_2} - C^{\dagger}_{MS}C)X_2 = (I_{H_2} - C^{\dagger}_{MS}C)Y_2. \tag{3.17}$$

The expression for U given by (3.5) follows from (3.13), (3.17) and (3.15) by letting $N_3 = S$. The conclusion then follows from (3.14). □

Theorem 3.3 below was proved in [1, 6, 10] for matrices by using different methods. In the context of Hilbert C^*-module operators, we can give a general proof as follows:

Theorem 3.3. *Under the conditions of Theorem 3.2 we have*

$$(A,B)^{\dagger}_{MN} = \begin{pmatrix} A^{\dagger}_{MN_1} - \Big(D + (I_{H_1} - A^{\dagger}_{MN_1}A)N_1^{-1}L\Big)\widetilde{U} \\ \widetilde{U} \end{pmatrix}, \tag{3.18}$$

where

$$D = A^{\dagger}_{MN_1}B \in \mathcal{L}(H_2, H_1), \tag{3.19}$$
$$\widetilde{S} = N_2 - L^*(I_{H_1} - A^{\dagger}_{MN_1}A)N_1^{-1}L + D^*N_1D - D^*L - L^*D \in \mathcal{L}(H_2), \tag{3.20}$$
$$\widetilde{U} = C^{\dagger}_{M\widetilde{S}} + (I_{H_2} - C^{\dagger}_{M\widetilde{S}}C)(\widetilde{S})^{-1}(D^*N_1 - L^*)A^{\dagger}_{MN_1} \in \mathcal{L}(H_3, H_2). \tag{3.21}$$

Proof. Let $T = \begin{pmatrix} I_{H_1} & -D \\ 0 & I_{H_2} \end{pmatrix} \in \mathcal{L}(H_1 \oplus H_2)$. Then T is invertible with $T^{-1} = \begin{pmatrix} I_{H_1} & D \\ 0 & I_{H_2} \end{pmatrix}$. In view of (3.2) and (3.19), we have

$$(A,B)\,T = (A,C), \tag{3.22}$$

which means that $\mathcal{R}(A,B) = \mathcal{R}(A,C)$ is closed, so $(A,B)^{\dagger}_{MN}$ exists. Furthermore, by (1.6) and (3.22) we know that $(A,B)^{\dagger}_{MN}$ is the unique solution $\widetilde{X} = \binom{\widetilde{X}_1}{\widetilde{X}_2} \in \mathcal{L}(H_3, H_1 \oplus H_2)$ to the equation

$$(A,C)^*M(A,C)T^{-1}\widetilde{X} = (A,C)^*M, \tag{3.23}$$
$$R\big(T^*NT \cdot T^{-1}\widetilde{X}\big) \subseteq R\Big((A,C)^*\Big). \tag{3.24}$$

It follows from (1.6) that $T^{-1}\widetilde{X} = (A,C)^{\dagger}_{M\widetilde{N}}$, where

$$\widetilde{N} = T^*NT = \begin{pmatrix} N_1 & L - N_1D \\ L^* - D^*N_1 & N_2 - D^*L - L^*D + D^*N_1D \end{pmatrix}. \tag{3.25}$$

By the definition of D we get $(I_{H_1} - A^\dagger_{MN_1}A)D = 0$ and

$$\begin{aligned}&D^*N_1(I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1} = B^*(A^\dagger_{MN_1})^*N_1(I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}\\ &= B^*(A^\dagger_{MN_1})^*(I_{H_1} - A^\dagger_{MN_1}A)^* = 0.\end{aligned}$$

In view of (3.4), if we replace N_2, L with $N_2 - D^*L - L^*D + D^*N_1D$ and $L - N_1D$ respectively, and define

$$\begin{aligned}\widetilde{S} &= (N_2 - D^*L - L^*D + D^*N_1D) - (L - N_1D)^*(I - A^\dagger_{MN_1}A)N_1^{-1}(L - N_1D)\\ &= N_2 - L^*(I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}L + D^*N_1D - D^*L - L^*D,\end{aligned}$$

then by Theorem 3.2 we conclude that $T^{-1}\widetilde{X} = \binom{V_1}{V_2}$ with

$$V_2 = C^\dagger_{M\widetilde{S}} - (I_{H_2} - C^\dagger_{M\widetilde{S}}C)(\widetilde{S})^{-1}(L - N_1D)^*A^\dagger_{MN_1}, \tag{3.26}$$

$$\begin{aligned}V_1 &= A^\dagger_{MN_1} - (I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}(L - N_1D)V_2\\ &= A^\dagger_{MN_1} - (I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}LV_2.\end{aligned} \tag{3.27}$$

As $\binom{\widetilde{X_1}}{\widetilde{X_2}} = T\binom{V_1}{V_2} = \binom{V_1 - DV_2}{V_2}$, (3.21) and (3.18) then follow from (3.26) and (3.27). □

Now we are ready to give a unified representation for $(A, B)^\dagger_{MN}$ in terms of $C^\dagger_{MN_3}$, where $N_3 \in \mathcal{L}(H_2)$ can be an arbitrary positive definite operator.

Theorem 3.4. *Under the conditions of Theorem 3.2 we have*

$$(A, B)^\dagger_{MN} = \begin{pmatrix} A^\dagger_{MN_1} - \big(D + (I_{H_1} - A^\dagger_{MN_1}A)N_1^{-1}L\big)V \\ V \end{pmatrix}, \tag{3.28}$$

where $N_3 \in \mathcal{L}(H_2)$ is arbitrary positive definite, D and $\widetilde{S}$ are defined by (3.19) and (3.20) respectively, and

$$\begin{aligned}&R_{M;N_3,\widetilde{S}} = I_{H_2} + (I_{H_2} - C^\dagger_{MN_3}C)N_3^{-1}(\widetilde{S} - N_3),\\ &V = R^{-1}_{M;N_3,\widetilde{S}}\Big(C^\dagger_{MN_3} + (I_{H_2} - C^\dagger_{MN_3}C)N_3^{-1}(D^*N_1 - L^*)A^\dagger_{MN_1}\Big).\end{aligned} \tag{3.29}$$

Proof. By Lemma 2.2 we have $C^\dagger_{M\widetilde{S}} = R^{-1}_{M;N_3,\widetilde{S}}C^\dagger_{MN_3}$. Furthermore, by (2.8) we can get

$$(I_{H_2} - C^\dagger_{M\widetilde{S}}C)\widetilde{S}^{-1} = R^{-1}_{M;N_3,\widetilde{S}} \cdot (I_{H_2} - C^\dagger_{MN_3}C)N_3^{-1}.$$

The conclusion then follows from (3.18) and (3.21). □

In the special case of the preceding theorem where $N_3 = S(N)$, we regain the main technique result of [11] as follows:

Theorem 3.5. (cf. [11, Theorem 5.1]) *Under the conditions of Theorem 3.2 we have*

$$(A, B)^\dagger_{MN} = \begin{pmatrix} A^\dagger_{MN_1} - (\Sigma + N_1^{-1}L)\Omega \\ \Omega \end{pmatrix}, \tag{3.30}$$

where C and D are defined by (3.2) and (3.19) respectively, and

$$\Sigma = A^\dagger_{MN_1}(B - AN_1^{-1}L) = D - A^\dagger_{MN_1}AN_1^{-1}L, \tag{3.31}$$

$$Y = \big(I - C^\dagger_{MS(N)}C\big)S(N)^{-1}, \tag{3.32}$$

$$\Omega = \big(I + Y\Sigma^* N_1 \Sigma\big)^{-1}\Big(Y\Sigma^* N_1 \cdot A^\dagger_{MN_1} + C^\dagger_{MS(N)}\Big). \tag{3.33}$$

Proof. Let $\widetilde{S}$ be given by (3.20) and define

$$\Delta = \widetilde{S} - S(N) = L^* A^\dagger_{MN_1} AN_1^{-1}L + D^* N_1 D - D^* L - L^* D. \tag{3.34}$$

By definition we have

$$\Sigma^* = D^* - L^* A^\dagger_{MN_1} AN_1^{-1}, \text{ so } \Sigma^* N_1 = D^* N_1 - L^* A^\dagger_{MN_1} A. \tag{3.35}$$

It follows that $\Sigma^* N_1 A^\dagger_{MN_1} = D^* N_1 A^\dagger_{MN_1} - L^* A^\dagger_{MN_1}$. Therefore,

$$(I - C^\dagger_{MS(N)}C)S(N)^{-1}(D^* N_1 - L^*)A^\dagger_{MN_1} = Y\Sigma^* N_1 A^\dagger_{MN_1}. \tag{3.36}$$

By the definition of D, we have $A^\dagger_{MN_1}AD = D$, so by (3.35) and (3.31) we have

$$\begin{aligned}
\Sigma^* N_1 \Sigma &= (D^* N_1 - L^* A^\dagger_{MN_1} A)(D - A^\dagger_{MN_1} AN_1^{-1}L) \\
&= D^* N_1 D - D^*(A^\dagger_{MN_1}A)^* L - L^* A^\dagger_{MN_1} AD + L^* A^\dagger_{MN_1} AN_1^{-1}L \\
&= D^* N_1 D - D^* L - L^* D + L^* A^\dagger_{MN_1} AN_1^{-1} L = \Delta.
\end{aligned} \tag{3.37}$$

It follows that

$$R_{M;S(N),\widetilde{S}} = I + Y\Delta = I + Y\Sigma^* N_1 \Sigma. \tag{3.38}$$

Finally, by the definitions of D and Σ we get

$$D + (I - A^\dagger_{MN_1}A)N_1^{-1}L = \Sigma + N_1^{-1}L. \tag{3.39}$$

Expression (3.33) for Ω follows from (3.29), (3.38) and (3.36). Formula (3.30) for $(A,B)^\dagger_{MN}$ then follows from (3.28) and (3.31). $\square$

4 Representations for weighted Moore-Penrose inverses of 2×2 partitioned operators

4.1 Non weighted case

Following the line initiated in [7], in this section we study the representations for the (non-weighted) Moore-Penrose inverse $A^\dagger$ of a general 2×2 partitioned operator matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathcal{L}(H_1 \oplus H_2, K_1 \oplus K_2), \tag{4.1}$$

where H_1, H_2, K_1 and K_2 are four Hilbert $\mathfrak{A}$-modules, $A_{11} \in \mathcal{L}(H_1, K_1)$, $A_{12} \in \mathcal{L}(H_2, K_1)$, $A_{21} \in \mathcal{L}(H_1, K_2)$ and $A_{22} \in \mathcal{L}(H_2, K_2)$. In the case when A_{11} has a closed range, let $S(A)$ be the Schur complement of A defined by

$$S(A) = A_{22} - A_{21}A_{11}^\dagger A_{12} \in \mathcal{L}(H_2, K_2). \tag{4.2}$$

4.1.1 Special case

Lemma 4.1. *Suppose that A_{11} has a closed range. Then both $F_1(A)^\dagger$ and $F_2(A)^\dagger$ exist, where*

$$F_1(A) = \begin{pmatrix} -A_{11}^\dagger A_{12} \\ I_{H_2} \end{pmatrix} \in \mathcal{L}(H_2, H_1 \oplus H_2), \tag{4.3}$$

$$F_2(A) = (-A_{21}A_{11}^\dagger, I_{K_2}) \in \mathcal{L}(K_1 \oplus K_2, K_2). \tag{4.4}$$

Furthermore, the following equalities hold:

(i) $F_1(A)^\dagger \cdot \begin{pmatrix} A_{11}^\dagger A_{11} & A_{11}^\dagger A_{12} \\ 0 & 0 \end{pmatrix} = F_1(A)^\dagger - (0, I_{H_2})$;

(ii) $\begin{pmatrix} A_{11}A_{11}^\dagger & 0 \\ A_{21}A_{11}^\dagger & 0 \end{pmatrix} \cdot F_2(A)^\dagger = F_2(A)^\dagger - \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix}$.

Proof. By definition we have $F_2(A)F_2(A)^* = I_{K_2} + \big(A_{21}A_{11}^\dagger\big)\big(A_{21}A_{11}^\dagger\big)^*$, which is invertible, hence by Proposition 1.6 we have

$$F_2(A)^\dagger = F_2(A)^* \cdot \big(F_2(A)F_2(A)^*\big)^{-1}. \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\begin{aligned} &\begin{pmatrix} A_{11}A_{11}^\dagger & 0 \\ A_{21}A_{11}^\dagger & 0 \end{pmatrix} F_2(A)^\dagger = \begin{pmatrix} -(A_{21}A_{11}^\dagger)^* \\ -(A_{21}A_{11}^\dagger)(A_{21}A_{11}^\dagger)^* \end{pmatrix} \big(F_2(A)F_2(A)^*\big)^{-1} \\ &= \left[F_2(A)^* - \begin{pmatrix} 0 \\ F_2(A)F_2(A)^* \end{pmatrix}\right] \big(F_2(A)F_2(A)^*\big)^{-1} = F_2(A)^\dagger - \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix}. \end{aligned}$$

The proof of (i) is similar. ∎

Theorem 4.2. (cf. [7, Theorem 2]) *Suppose that both A_{11} and $S(A)$ have closed ranges, and*

$$(I_{K_1} - A_{11}A_{11}^\dagger)A_{12} = 0, \ \ A_{21}(I_{H_1} - A_{11}^\dagger A_{11}) = 0. \tag{4.6}$$

Then

$$A^\dagger = X_L(A)\,\mathrm{diag}(A_{11}^\dagger, 0)\,X_R(A) + F_1(A)\,S(A)^g\,F_2(A), \tag{4.7}$$

where $F_1(A)$ and $F_2(A)$ are defined by (4.3) and (4.4) respectively, and

$$S(A)^g = S(A)^\dagger_{[F_2(A)F_2(A)^*]^{-1}, F_1(A)^*F_1(A)} \in \mathcal{L}(K_2, H_2), \tag{4.8}$$

$$X_L(A) = I_{H_1\oplus H_2} - F_1(A)\big[I_{H_2} - S(A)^g S(A)\big]F_1(A)^\dagger \in \mathcal{L}(H_1 \oplus H_2), \tag{4.9}$$

$$X_R(A) = I_{K_1\oplus K_2} - F_2(A)^\dagger\big[I_{K_2} - S(A)S(A)^g\big]F_2(A) \in \mathcal{L}(K_1 \oplus K_2). \tag{4.10}$$

Proof. It follows from (4.1), (4.3), (4.4) and (4.6) that

$$AF_1(A) = \begin{pmatrix} 0 \\ S(A) \end{pmatrix} \text{ and } \ F_2(A)A = \big(0, S(A)\big), \tag{4.11}$$

which implies that

$$AX_L(A) = A \text{ and } X_R(A)A = A. \tag{4.12}$$

To simplify the notation, let

$$\lambda_1(A) = I_{H_2} - S(A)^g S(A) \text{ and } \lambda_2(A) = I_{K_2} - S(A)S(A)^g. \tag{4.13}$$

Then by (ii) of Lemma 4.1 we have

$$\begin{aligned}
&A\,\mathrm{diag}(A_{11}^\dagger, 0)\, X_R(A) = \begin{pmatrix} A_{11}A_{11}^\dagger & 0 \\ A_{21}A_{11}^\dagger & 0 \end{pmatrix} X_R(A) \\
&= \begin{pmatrix} A_{11}A_{11}^\dagger & 0 \\ A_{21}A_{11}^\dagger & 0 \end{pmatrix} - \begin{pmatrix} A_{11}A_{11}^\dagger & 0 \\ A_{21}A_{11}^\dagger & 0 \end{pmatrix} F_2(A)^\dagger \lambda_2(A) F_2(A) \\
&= \begin{pmatrix} A_{11}A_{11}^\dagger & 0 \\ A_{21}A_{11}^\dagger & 0 \end{pmatrix} - F_2(A)^\dagger \lambda_2(A) F_2(A) + \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} \lambda_2(A) F_2(A).
\end{aligned} \tag{4.14}$$

Furthermore, by the first equality in (4.11) we get

$$\begin{aligned}
&AF_1(A)S(A)^g F_2(A) = \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} S(A)S(A)^g F_2(A) \\
&= \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} \big(I_{K_2} - \lambda_2(A)\big) F_2(A) = \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} F_2(A) - \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} \lambda_2(A) F_2(A) \\
&= \begin{pmatrix} 0 & 0 \\ -A_{21}A_{11}^\dagger & I_{K_2} \end{pmatrix} - \begin{pmatrix} 0 \\ I_{K_2} \end{pmatrix} \lambda_2(A) F_2(A).
\end{aligned} \tag{4.15}$$

Now let Z be the right side of (4.7). Then by the first equality in (4.12), (4.14) and (4.15), we get

$$AZ = \mathrm{diag}(A_{11}A_{11}^\dagger, I_{K_2}) - F_2(A)^*\big(F_2(A)F_2(A)^*\big)^{-1}\lambda_2(A)F_2(A), \tag{4.16}$$

which means that $(AZ)^* = AZ$, since by the definitions of $S(A)^g$ and $\lambda_2(A)$ we have

$$\lambda_2(A)^* = \left(F_2(A)F_2(A)^*\right)^{-1} \lambda_2(A)\big(F_2(A)F_2(A)^*\big).$$

As $\lambda_2(A)\big(0, S(A)\big) = 0$, we may combine (4.16) with the second equality in (4.11) to get

$$AZA = \mathrm{diag}(A_{11}A_{11}^\dagger, I_{K_2})A = A.$$

Similarly, as $F_1(A)^\dagger = \big(F_1(A)^*F_1(A)\big)^{-1}F_1(A)^*$ and

$$\mathrm{diag}(A_{11}^\dagger A_{11}, I_{H_2})X_L(A) = X_L(A) - \mathrm{diag}\big(I_{H_1} - A_{11}^\dagger A_{11}, 0\big),$$

we can prove that

$$ZA = \mathrm{diag}(A_{11}^\dagger A_{11}, I_{H_2}) - F_1(A)\lambda_1(A)\big(F_1(A)^*F_1(A)\big)^{-1}F_1(A)^*$$

with $(ZA)^* = ZA$ and $ZAZ = Z$, therefore $Z = A^\dagger$. $\square$

Corollary 4.3. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \in \mathcal{L}(K_1 \oplus K_2)$ *be positive, where* $A_{ij} \in \mathcal{L}(K_j, K_i)(i,j = 1,2)$. *If both* $\mathcal{R}(A_{11})$ *and* $\mathcal{R}\big(S(A)\big)$ *are closed, then*

$$A^\dagger = X_L(A)\,\mathrm{diag}(A_{11}^\dagger, 0)\,X_R(A) + F_1(A)\,S(A)^g\,F_1(A)^*, \tag{4.17}$$

where $F_1(A)$ *is defined by (4.3),* $S(A)^g$, $X_L(A)$ *and* $X_R(A)$ *are given respectively as (4.8), (4.9) and (4.10) by letting* $F_2(A)$ *be replaced with* $F_1(A)^*$. *In addition, a* $\{1,3\}$*-inverse of* A *can be given by*

$$A^{(1,3)} = \mathrm{diag}(A_{11}^\dagger, 0)\,X_R(A) + F_1(A)\,S(A)^g\,F_1(A)^*. \tag{4.18}$$

Proof. Since A is positive, by [13, Corollary 3.5] we have

$$A_{11} \geq 0,\ A_{12} = A_{11}A_{11}^\dagger A_{12} \text{ and } S(A) \geq 0. \tag{4.19}$$

As $(A_{11}^\dagger)^* = A_{11}^\dagger$, conditions in (4.6) are satisfied. Note that in this case $F_2(A) = \big(F_1(A)\big)^*$, (4.17) follows from (4.7). Let $A^{(1,3)}$ be the operator given by (4.18). As $AX_L(A) = A$ we have $AA^{(1,3)} = AA^\dagger$, so $A^{(1,3)}$ is a $\{1,3\}$-inverse of A. $\square$

4.1.2 General case

Let

$$E = AA^* \overset{def}{=} \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^* & E_{22} \end{pmatrix} \in \mathcal{L}(K_1 \oplus K_2). \tag{4.20}$$

If (A_{11}, A_{12}) has a closed range, then as $E_{11} = (A_{11}, A_{12})(A_{11}, A_{12})^*$, by Lemma 1.1 and Proposition 1.6 we know that $E_{11}^\dagger$ exists such that $(A_{11}, A_{12})^\dagger = (A_{11}, A_{12})^* E_{11}^\dagger$. Let $S(E) = E_{22} - E_{12}^* E_{11}^\dagger E_{12}$ be the Schur complement of E. Assuming further that both A and $S(E)$ have closed ranges, then for any $\{1,3\}$-inverse $E^{(1,3)}$ of E, we have $A^\dagger = A^* E^{(1,3)}$. In particular, by (4.18) we have

$$A^\dagger = A^* \cdot \Big[\mathrm{diag}(E_{11}^\dagger, 0)\,X_R(E) + F_1(E)\,S(E)^g\,F_1(E)^*\Big], \tag{4.21}$$

where

$$F_1(E) = \begin{pmatrix} -E_{11}^\dagger E_{12} \\ I_{K_2} \end{pmatrix} \in \mathcal{L}(K_2, K_1 \oplus K_2), \tag{4.22}$$

$$S(E)^g = S(E)^\dagger_{[F_1(E)^*F_1(E)]^{-1}, F_1(E)^*F_1(E)} \in \mathcal{L}(K_2), \tag{4.23}$$

$$X_R(E) = I_{K_1 \oplus K_2} - \big(F_1(E)^*\big)^\dagger \big(I_{K_2} - S(E)S(E)^g\big)F_1(E)^* \in \mathcal{L}(K_1 \oplus K_2). \tag{4.24}$$

4.2 The weighted case

Following the line initiated in [11] for 1×2 partitioned operators, in this subsection we provide an approach to the construction of Moore-Penrose inverses of 2×2 partitioned operators from the non-weighted case to the weighted case. A detailed description of our idea can be illustrated as follows.

For any Hilbert $\mathfrak{A}$-module X, and any projection P of $\mathcal{L}(X)$, let $X_1 = PX$ and $X_2 = (I_X - P)X$, and define $\lambda_X : X \to X_1 \oplus X_2$ by

$$\lambda_X(x) = \begin{pmatrix} Px \\ x - Px \end{pmatrix}, \text{ for any } x \in X. \tag{4.25}$$

Then λ_X is a unitary operator with $\lambda_X^* = \lambda_X^{-1}$, where $\lambda_X^{-1} : X_1 \oplus X_2 \to X$ is given by

$$\lambda_X^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + x_2, \text{ for any } x_i \in X_i, i = 1, 2.$$

Now let H_1 and H_2 be two Hilbert $\mathfrak{A}$-modules,

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{12}^* & N_{22} \end{pmatrix} \in \mathcal{L}(H_1 \oplus H_2) \tag{4.26}$$

be a positive definite operator, where $N_{11} \in \mathcal{L}(H_1)$, $N_{12} \in \mathcal{L}(H_2, H_1)$ and $N_{22} \in \mathcal{L}(H_2)$. Let $S(N) = N_{22} - N_{12}^* N_{11}^{-1} N_{12}$ be the Schur complement of N. Define

$$a = N_{11}^{-1} N_{12},\ P = \begin{pmatrix} I_{H_1} & a \\ 0 & 0 \end{pmatrix} \text{ and } \quad X = (H_1 \oplus H_2)_N. \tag{4.27}$$

Then $P^2 = P$ and $NP = P^* N$, so $P^\# = N^{-1} P^* N = P$, which means that $P \in \mathcal{L}(X)$ is a projection of $\mathcal{L}(X)$, where X is the weighted space define by (4.27) whose inner-product is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_N = \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, N\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, N_{11}x_2 + N_{12}y_2 \rangle + \langle y_1, N_{12}^* x_2 + N_{22} y_2 \rangle$$

for any $x_i \in H_1$ and $y_i \in H_2, i = 1, 2$. By (4.27) we have

$$X_1 = PX = \left\{ \begin{pmatrix} h_1 + ah_2 \\ 0 \end{pmatrix} \middle| h_i \in H_i \right\} = \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \middle| u \in H_1 \right\}, \tag{4.28}$$

$$X_2 = (I_X - P)X = \left\{ \begin{pmatrix} -ah_2 \\ h_2 \end{pmatrix} \middle| h_2 \in H_2 \right\}. \tag{4.29}$$

With the inner products inherited from X, both X_1 and X_2 are Hilbert $\mathfrak{A}$-modules. Let $j_{H_1} : (H_1)_{N_{11}} \to X_1$ and $j_{H_2} : (H_2)_{S(N)} \to X_2$ be defined by

$$j_{H_1}(h_1) = \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \text{ and } j_{H_2}(h_2) = \begin{pmatrix} -ah_2 \\ h_2 \end{pmatrix}, \text{ for any } h_i \in H_i, i = 1, 2.$$

It is easy to verify that both j_{H_1} and j_{H_2} are unitary operators with

$$j_{H_1}^{-1}\begin{pmatrix} h_1 \\ 0 \end{pmatrix} = h_1 \text{ and } j_{H_2}^{-1}\begin{pmatrix} -ah_2 \\ h_2 \end{pmatrix} = h_2, \text{ for any } h_i \in H_i, i = 1, 2.$$

Let $j_{H_1} \oplus j_{H_2} : (H_1)_{N_{11}} \oplus (H_2)_{S(N)} \to X_1 \oplus X_2$ be the associated unitary operator defined by

$$(j_{H_1} \oplus j_{H_2})\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} j_{H_1}(h_1) \\ j_{H_2}(h_2) \end{pmatrix} = \begin{pmatrix} h_1 \\ 0 \\ --- \\ -ah_2 \\ h_2 \end{pmatrix}, \text{ for any } h_i \in H_i, i = 1, 2.$$

Then clearly, $(j_{H_1}\oplus j_{H_2})^{\#}=(j_{H_1}\oplus j_{H_2})^{-1}=j_{H_1}^{-1}\oplus j_{H_2}^{-1}=j_{H_1}^{\#}\oplus j_{H_2}^{\#}$.

Now suppose that K_1 and K_2 are two additional Hilbert $\mathfrak{A}$-modules, and

$$M=\begin{pmatrix} M_{11} & M_{12}\\ M_{12}^* & M_{22}\end{pmatrix}\in\mathcal{L}(K_1\oplus K_2) \tag{4.30}$$

is a positive definite operator, where $M_{11}\in\mathcal{L}(K_1)$, $M_{12}\in\mathcal{L}(K_2,K_1)$ and $M_{22}\in\mathcal{L}(K_2)$. Let $S(M)=M_{22}-M_{12}^*M_{11}^{-1}M_{12}$ be the Schur complement of M, and define

$$b=M_{11}^{-1}M_{12},\ Q=\begin{pmatrix} I_{K_1} & b\\ 0 & 0\end{pmatrix}\ \text{ and }\quad Y=(K_1\oplus K_2)_M. \tag{4.31}$$

Similarly, define $Y_1=QY, Y_2=(I_Y-Q)Y, \lambda_Y:Y\to Y_1\oplus Y_2$, $j_{K_1}:(K_1)_{M_{11}}\to Y_1$ and $j_{K_2}:(K_2)_{S(M)}\to Y_2$.

With the notation as above and suppose further that

$$A=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\in\mathcal{L}(H_1\oplus H_2,K_1\oplus K_2),$$

where $A_{11}\in\mathcal{L}(H_1,K_1), A_{12}\in\mathcal{L}(H_2,K_1), A_{21}\in\mathcal{L}(H_1,K_2)$ and $A_{22}\in\mathcal{L}(H_2,K_2)$. Then we have the following commutative diagram:

$$\begin{array}{ccccc}
(H_1)_{N_{11}}\oplus(H_2)_{S(N)} & \xrightarrow{j_{H_1}\oplus j_{H_2}} & X_1\oplus X_2 & \xrightarrow{\lambda_X^{-1}} & X=(H_1\oplus H_2)_N\\
\Big\downarrow{\scriptstyle B} & & & & \Big\downarrow{\scriptstyle A}\\
(K_1)_{M_{11}}\oplus(K_2)_{S(M)} & \xrightarrow{j_{K_1}\oplus j_{K_2}} & Y_1\oplus Y_2 & \xrightarrow{\lambda_Y^{-1}} & Y=(K_1\oplus K_2)_M
\end{array}$$

where

$$B=\begin{pmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix}=(j_{K_1}^{-1}\oplus j_{K_2}^{-1})\circ\lambda_Y\circ A\circ\lambda_X^{-1}\circ(j_{H_1}\oplus j_{H_2}), \tag{4.32}$$

with

$$B_{11}=A_{11}+M_{11}^{-1}M_{12}A_{21}, \tag{4.33}$$
$$B_{12}=A_{12}+M_{11}^{-1}M_{12}A_{22}-A_{11}N_{11}^{-1}N_{12}-M_{11}^{-1}M_{12}A_{21}N_{11}^{-1}N_{12}, \tag{4.34}$$
$$B_{21}=A_{21}, \tag{4.35}$$
$$B_{22}=A_{22}-A_{21}N_{11}^{-1}N_{12}. \tag{4.36}$$

Since $\lambda_X,\lambda_Y,j_{H_1}\oplus j_{H_2}$ and $j_{K_1}\oplus j_{K_2}$ are all unitary operators, by (4.32) we get

$$A_{MN}^\dagger=\lambda_X^{-1}\circ(j_{H_1}\oplus j_{H_2})\circ B^\dagger_{\mathrm{diag}\big(M_{11},S(M)\big),\mathrm{diag}\big(N_{11},S(N)\big)}\circ(j_{K_1}^{-1}\oplus j_{K_2}^{-1})\circ\lambda_Y. \tag{4.37}$$

So if we let

$$B^\dagger_{\mathrm{diag}\big(M_{11},S(M)\big),\mathrm{diag}\big(N_{11},S(N)\big)}=\begin{pmatrix} (B^\dagger)_{11} & (B^\dagger)_{12}\\ (B^\dagger)_{21} & (B^\dagger)_{22}\end{pmatrix},$$

where

$$(B^\dagger)_{11}\in\mathcal{L}\big((K_1)_{M_{11}},(H_1)_{N_{11}}\big),\quad (B^\dagger)_{12}\in\mathcal{L}\big((K_2)_{S(M)},(H_1)_{N_{11}}\big),$$
$$(B^\dagger)_{21}\in\mathcal{L}\big((K_1)_{M_{11}},(H_2)_{S(N)}\big),\ (B^\dagger)_{22}\in\mathcal{L}\big((K_2)_{S(M)},(H_2)_{S(N)}\big),$$

then by (4.37) we conclude that $A^\dagger_{MN} = \begin{pmatrix} (A^\dagger_{MN})_{11} & (A^\dagger_{MN})_{12} \\ (A^\dagger_{MN})_{21} & (A^\dagger_{MN})_{22} \end{pmatrix}$ with $(A^\dagger_{MN})_{11} \in \mathcal{L}(K_1, H_1)$, $(A^\dagger_{MN})_{12} \in \mathcal{L}(K_2, H_1)$, $(A^\dagger_{MN})_{21} \in \mathcal{L}(K_1, H_2)$ and $(A^\dagger_{MN})_{22} \in \mathcal{L}(K_2, H_2)$, such that

$$(A^\dagger_{MN})_{11} = (B^\dagger)_{11} - N_{11}^{-1} N_{12} (B^\dagger)_{21}, \tag{4.38}$$

$$(A^\dagger_{MN})_{12} = (B^\dagger)_{11} M_{11}^{-1} M_{12} + (B^\dagger)_{12} - N_{11}^{-1} N_{12} (B^\dagger)_{21} M_{11}^{-1} M_{12} - N_{11}^{-1} N_{12} (B^\dagger)_{22}, \tag{4.39}$$

$$(A^\dagger_{MN})_{21} = (B^\dagger)_{21}, \tag{4.40}$$

$$(A^\dagger_{MN})_{22} = (B^\dagger)_{21} M_{11}^{-1} M_{12} + (B^\dagger)_{22}. \tag{4.41}$$

Note that $(H_1)_{N_{11}}, (H_2)_{S(N)}, (K_1)_{M_{11}}$ and $(K_2)_{S(M)}$ are all Hilbert $\mathfrak{A}$-modules, the Moore-Penrose inverse of $B_{11} \in \mathcal{L}\big((H_1)_{N_{11}}, (K_1)_{M_{11}}\big)$ equals $(B_{11})^\dagger_{M_{11}, N_{11}}$, and the adjoint operator $B_{11}^\#$ of $B_{11} \in \mathcal{L}\big((H_1)_{N_{11}}, (K_1)_{M_{11}}\big)$ equals $N_{11}^{-1} B_{11}^* M_{11} \in \mathcal{L}(K_1, H_1)$. Since formula (4.21) is valid for any Hilbert $\mathfrak{A}$-module operators, we may use this formula to get a concrete expression for $B^\dagger_{\mathrm{diag}\left(M_{11}, S(M)\right), \mathrm{diag}\left(N_{11}, S(N)\right)}$, and then obtain an expression for $A^\dagger_{MN}$ by (4.38)–(4.41).

5 A numerical example

Example 5.1. Let $M = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with

$$A_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_{12} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, A_{21} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \text{ and } A_{22} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Then $M_{11} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $N_{11} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $S(M) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$ and $S(N) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$. By (4.33)–(4.36) we have

$$B_{11} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, B_{12} = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}, B_{21} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, B_{22} = \begin{pmatrix} -\frac{2}{3} & 2 \\ 0 & 0 \end{pmatrix}.$$

Note that the matrix $B = (B_{ij})_{1 \le i,j \le 2}$, regarded as an element of

$$\begin{aligned} &\mathcal{L}\left((H_1)_{N_{11}} \oplus (H_2)_{S(N)}, (K_1)_{M_{11}} \oplus (K_2)_{S(M)}\right) \\ &= \mathcal{L}\left((H_1 \oplus H_2)_{\mathrm{diag}\left(N_{11}, S(N)\right)}, (K_1 \oplus K_2)_{\mathrm{diag}\left(M_{11}, S(M)\right)}\right), \end{aligned}$$

whose conjugate $B^\#$ is given by

$$B^\# = \mathrm{diag}\big(N_{11}, S(N)\big)^{-1} \cdot B^* \cdot \mathrm{diag}\big(M_{11}, S(M)\big) = \begin{pmatrix} 2 & 0 & \frac{1}{3} & 0 \\ -2 & 0 & -\frac{2}{3} & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 3 & 1 & 0 \end{pmatrix}.$$

Let $E = BB^{\#} = \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^* & E_{22} \end{pmatrix} \in \mathcal{L}\left((K_1)_{M_{11}} \oplus (K_2)_{S(M)}\right)$, where

$$E_{11} = \begin{pmatrix} 4 & 0 \\ 0 & 12 \end{pmatrix}, E_{12} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} \text{ and } E_{22} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

By direct computation we have

$$\begin{aligned}
&(E_{11})^{\dagger}_{M_{11},M_{11}} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{12} \end{pmatrix}, F_1(E) = \begin{pmatrix} -(E_{11})^{\dagger}_{M_{11},M_{11}} E_{12} \\ I_{K_2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & 0 \\ -\frac{1}{6} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
&F_1(E)^{\#} = S(M)^{-1} \cdot F_1(E)^* \cdot \operatorname{diag}(M_{11}, S(M)) = \begin{pmatrix} -1 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
&F_1(E)^{\#} \cdot F_1(E) = \begin{pmatrix} \frac{47}{36} & 0 \\ 0 & 1 \end{pmatrix}, S(E) = E_{22} - E_{21} \cdot (E_{11})^{\dagger}_{M_{11},M_{11}} \cdot E_{12} = \begin{pmatrix} \frac{7}{3} & 0 \\ 0 & 0 \end{pmatrix}, \\
&Z_1 \overset{def}{=} S(M) F_1(E)^{\#} F_1(E) = \begin{pmatrix} \frac{47}{72} & 0 \\ 0 & 1 \end{pmatrix}, Z_2 \overset{def}{=} S(M)\big(F_1(E)^{\#} F_1(E)\big)^{-1} = \begin{pmatrix} \frac{18}{47} & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Note that $\big((K_2)_{S(M)}\big)_{F_1(E)^{\#}F_1(E)} = (K_2)_{Z_1}$ and $\big((K_2)_{S(M)}\big)_{\left(F_1(E)^{\#}F_1(E)\right)^{-1}} = (K_2)_{Z_2}$, so by (4.23) we have

$$S(E)^g = S(E)^{\dagger}_{Z_2,Z_1} = \begin{pmatrix} \frac{3}{7} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $T = \begin{pmatrix} -1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix} \in \mathcal{L}\left((K_1)_{M_{11}}, (K_2)_{S(M)}\right)$ and $Z_3 = \operatorname{diag}\big(M_{11}, S(M)\big)$. As

$$F_1(E)^{\#} = (T, I_{K_2}) \in \mathcal{L}\left((K_1)_{M_{11}} \oplus (K_2)_{S(M)}, (K_2)_{S(M)}\right) = \mathcal{L}\left((K_1 \oplus K_2)_{Z_3}, (K_2)_{S(M)}\right),$$

if we replace $H_1, H_2, H_3, A, B, M, N_1, L$ and N_2 with $K_1, K_2, K_2, T, I_{K_2}, S(M), M_{11}, 0$ and $S(M)$ respectively, then we may apply Theorem 3.3 to get

$$\big(F_1(E)^{\#}\big)^{\dagger}_{S(M),Z_3} = \begin{pmatrix} T^{\dagger}_{S(M),M_{11}} - D\widetilde{U} \\ \widetilde{U} \end{pmatrix},$$

where

$$\begin{aligned}
&D = T^{\dagger}_{S(M),M_{11}} = \begin{pmatrix} -\frac{9}{11} & 0 \\ -\frac{6}{11} & 0 \end{pmatrix}, \ \widetilde{S} = S(M) + D^* M_{11} D = \begin{pmatrix} \frac{47}{22} & 0 \\ 0 & 1 \end{pmatrix}, \\
&C = I_{K_2} - TT^{\dagger}_{S(M),M_{11}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ C^{\dagger}_{S(M),\widetilde{S}} = C, \\
&\widetilde{U} = C^{\dagger}_{S(M),\widetilde{S}} + \left(I_{K_2} - C^{\dagger}_{S(M),\widetilde{S}} C\right) (\widetilde{S})^{-1} D^* M_{11} T^{\dagger}_{S(M),M_{11}} = \begin{pmatrix} \frac{36}{47} & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\left(F_1(E)^{\#}\right)^{\dagger}_{S(M),Z_3} = \begin{pmatrix} -\frac{9}{47} & 0 \\ -\frac{6}{47} & 0 \\ \frac{36}{47} & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows from (4.24) that

$$X_R(E) = I_{K_1 \oplus K_2} - \left(F_1(E)^{\#}\right)^{\dagger}_{S(M),Z_3}\left(I_{K_2} - S(E)S(E)^g\right)F_1(E)^{\#} = \mathrm{diag}(1,1,1,0),$$

hence by (4.21) we get

$$\begin{aligned} & B^{\dagger}_{\mathrm{diag}\left(M_{11},S(M)\right),\mathrm{diag}\left(N_{11},S(N)\right)} \\ &= B^{\#}\cdot\left[\mathrm{diag}\left((E_{11})^{\dagger}_{M_{11},M_{11}},0\right)\, X_R(E) + F_1(E)\,S(E)^g\,F_1(E)^{\#}\right] = \begin{pmatrix} (B^{\dagger})_{11} & (B^{\dagger})_{12} \\ (B^{\dagger})_{21} & (B^{\dagger})_{22} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} (B^{\dagger})_{11} &= \begin{pmatrix} \frac{4}{7} & \frac{1}{42} \\ -\frac{3}{7} & \frac{1}{42} \end{pmatrix}, \quad (B^{\dagger})_{12} = \begin{pmatrix} -\frac{1}{14} & 0 \\ -\frac{1}{14} & 0 \end{pmatrix}, \\ (B^{\dagger})_{21} &= \begin{pmatrix} \frac{9}{14} & \frac{13}{28} \\ -\frac{3}{14} & \frac{5}{28} \end{pmatrix}, \quad (B^{\dagger})_{22} = \begin{pmatrix} -\frac{9}{14} & 0 \\ \frac{3}{14} & 0 \end{pmatrix}. \end{aligned}$$

It follows from (4.38)–(4.41) that

$$\begin{aligned} (A^{\dagger}_{MN})_{11} &= \begin{pmatrix} \frac{1}{7} & -\frac{2}{7} \\ -\frac{3}{14} & \frac{5}{28} \end{pmatrix}, \quad (A^{\dagger}_{MN})_{12} = \begin{pmatrix} \frac{3}{7} & 0 \\ -\frac{11}{28} & 0 \end{pmatrix}, \\ (A^{\dagger}_{MN})_{21} &= \begin{pmatrix} \frac{9}{14} & \frac{13}{28} \\ -\frac{3}{14} & \frac{5}{28} \end{pmatrix}, \quad (A^{\dagger}_{MN})_{22} = \begin{pmatrix} -\frac{9}{28} & 0 \\ \frac{3}{28} & 0 \end{pmatrix}, \end{aligned}$$

therefore,

$$A^{\dagger}_{MN} = \begin{pmatrix} \frac{1}{7} & -\frac{2}{7} & \frac{3}{7} & 0 \\ -\frac{3}{14} & \frac{5}{28} & -\frac{11}{28} & 0 \\ \frac{9}{14} & \frac{13}{28} & -\frac{9}{28} & 0 \\ -\frac{3}{14} & \frac{5}{28} & \frac{3}{28} & 0 \end{pmatrix}.$$

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